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Magneto-Fluid Dynamics Division

Energy Principle With Specific Inductance

Glenn Bateman

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Erratum

"Energy Principle With Specific Inductance"

By Glenn Bateman

Page 16 bottom: Delete "In particular, we see from the ideal MHD equation of motion..." to end of paragraph.

Replace with: "For this minimization we must demand that each element of the plasma is able to communicate with the rest of the flux surface. In particular, we exclude from this theory those systems in which all the field lines are closed."

Page 17: the equation after Eq. (28): Delete "2π".

Page 20 top: Delete "The variation of flux and inductance separately..." to the end of paragraph.

Page 23 top: Delete "However, as the shells change shape,..." to end of paragraph. Replace with: "It can be shown that individual fluid elements also do not compress or expand during their motion within a shell as they reach equilibrium."

Page 35 end of first sentence: Delete: "...before comparing states with a variational analysis."
Replace with: "...in successive stages within the class of accessible states."

the specific inductance matrix. The latter depends upon only the geometry of the surfaces. Stability conditions are derived by varying the flux profile and the geometry (specific inductance). Also, a nonlinear condition is derived for the special case of finite geometric distortions of the flux surfaces holding the flux profiles fixed. Exact results for the specific inductance are calculated for straight cylinders, axisymmetric toruses, and helically symmetric configurations, all with arbitrary cross sections.

Abstract

The determination of stability for a scalar pressure, toroidal plasma has been formulated by means of an energy variational principle in which the geometric problem of variation of the flux surface shapes is separated from the variation of the flux profile. By means of successive minimizations, holding the surface shapes and the flux profiles fixed, the potential energy is written as a one-dimensional integral over a function of pressure, differential flux, and the specific inductance matrix. The latter depends upon only the geometry of the surfaces. Stability conditions are derived by varying the flux profile and the geometry (specific inductance). Also, a nonlinear condition is derived for the special case of finite geometric distortions of the flux surfaces holding the flux profiles fixed. Exact results for the specific inductance are calculated for straight cylinders, axisymmetric toruses, and helically symmetric configurations, all with arbitrary cross sections.

I. Introduction

The energy principle for a scalar pressure plasma can be reduced, after auxiliary minimizations, to a form involving the geometry only through a specific inductance matrix.[1]

The motivation for using the magnetic inductance is illustrated by the following model. Consider the vacuum magnetic field enclosed between two nested toroidal conductors. Blank, Friedrichs and Grad[2] show that the magnetic energy can be written as the product of the total currents and the total magnetic fluxes (Hodges' theorem). Furthermore, the fluxes can be written as a linear combination of the total currents.[2,3,4] The factors of proportionality, called the inductance matrix, depend upon only the geometry of the conductors and their relative position. Therefore, the behavior of the energy can be investigated by separately varying the geometry (inductance) or the total currents. Once the inductance is known, the energy can be calculated from the total currents without first calculating the distribution of current density on the conductors or computing the magnetic field.

In this paper, the above ideas are modified to apply to toroidal systems (plasmas) with distributed current. Given a set of simply nested toroidal flux surfaces and a pair of total azimuthal and longitudinal currents for each surface, there exists one magnetic field configuration which minimizes the magnetic energy on each surface.[5,6] For this magnetic field, we show that the total currents can be written as a linear combination of the fluxes trapped between differentially close flux surfaces. The four coefficients in the linear combination are called the specific inductance matrix. For each flux surface, the specific inductance depends upon only the geometry of the surface and its geometric relation to differentially close neighboring surfaces. This is proved in section II-C. Grad and Rubin[5] and Kruskal and Kulsrud[6] show that this minimized magnetic energy reduces to a one dimensional integral over the product of total currents and differential fluxes. Using the specific inductance, the integrand is factored into geometric and profile dependent matricies.[1] We may then vary the energy by varying the geometry, holding the flux profile fixed as a function of volume, or by varying the flux profile while holding the geometry of the surfaces fixed.

If we use an ideal magnetohydrodynamic model and minimize the magnetic and thermodynamic energy on each surface, the entire potential energy reduces to a one dimensional integration. If the appropriate constraints are used to assure that states are accessible to each other by a possible motion of the plasma, it can be shown[5,6] that the state of extremum potential energy corresponds to the classical definition of static equilibrium. We call the system unstable if the potential energy, with the magnetic and thermodynamic energy minimized on each surface, is lower in any other accessible state than in the equilibrium being considered. In section III-E we show that this definition corresponds to the definition used in the classical energy principle.[7]

In order to be able to use the variational formulations of stability derived in sections III-B, C, and D, we must be able to calculate the specific inductance for both the equilibrium and all the perturbed states to be considered. In section II-C, a number of formal representations for the specific inductance are derived. In section IV, explicit results are found (in terms of one integration) for the specific inductance of straight cylindrical, axisymmetric toroidal, and helically symmetric surfaces, all with arbitrary cross sections. The integration can be performed analytically in the first two cases when the cross sections are circular with arbitrarily displaced centers.

II. Magnetic Energy in Terms of Specific Inductance

A. Flux Surfaces

In this section we shall characterize the class of magnetic fields which are compatible with a given set of flux surfaces. Setting aside those problems associated with closed field lines (such as island structure), we shall assume that these surfaces exist and are nested in a single set around one magnetic axis. Each surface is labeled by its volume V and is represented by a scalar function $V = V(\theta_1, \theta_2, r)$. When an explicit representation is needed, we use a right handed coordinate system (θ_1, θ_2, r) with the same topology as the flux surfaces as illustrated in Fig. 1; θ_1 and θ_2 increase by unity the long and the short way around the torus respectively.

On each toroidal surface there are two homology classes of closed line contours and *cut* surfaces. We use C_1 and C_2 to represent closed contours the long and the short way around respectively and S_1 and S_2 to represent a *reciprocal* basis of cut surfaces which are bounded by contours $-C_2$ and $+C_1$ respectively (note the exchange of subscripts). These conventions are illustrated in Fig. 1 for toroidal surfaces but they may also be applied to cylindrical surfaces with periodic end conditions. Any closed contour on the torus is homologous to integer multiples of the representative contours C_1 and C_2. Blank, Friedrichs and Grad[1] discuss these concepts in more generality.

The magnetic flux through any cut surface S_i is defined by

$$\psi_i \equiv \int_{S_i} d\underset{\sim}{s} \cdot \underset{\sim}{B} \ . \tag{1}$$

From Fig. 1 we see that ψ_1 is the longitudinal flux within the flux surface while ψ_2 is the azimuthal flux outside the surface. A flux surface S_V is characterized by the property that each flux is independent of the choice among homologous cut surfaces. Using only the divergence-free property of magnetic fields

$$\nabla \cdot \underset{\sim}{B} = 0 \tag{2}$$

it can be shown that those surfaces which are ergodically covered by non-closed field lines are in fact flux surfaces. The total current through any closed contour C_i is defined by Ampere's law

$$I_i \equiv \oint_{C_i} d\underset{\sim}{\ell} \cdot \underset{\sim}{B} \ . \tag{3}$$

Here I_1 is the total azimuthal current outside and I_2 is the total longitudinal current within a current surface. A current surface has the property that each total current is independent of the choice among homologous closed contours on that surface.

Any magnetic field with a given set of flux surfaces may be characterized by a scalar potential in the following way. From Eq. (2) and the divergence theorem it can be shown that the magnetic field lines must everywhere lie in the flux surfaces

$$\underset{\sim}{B}\cdot\nabla V = 0 \ . \tag{4}$$

Equation (4) implies that there exists a vector field $\underset{\sim}{w}$ such that

$$\underset{\sim}{B} = \nabla V \times \underset{\sim}{w} \ .$$

When this characterization of the magnetic field is inserted into Equation (2) we find

$$\nabla V\cdot\nabla \times \underset{\sim}{w} = 0 \ . \tag{5}$$

Without loss of generality we may omit the normal component of $\underset{\sim}{w}$

$$\underset{\sim}{w}\cdot\nabla V = 0 \tag{6}$$

It can be shown that any vector field satisfying Equations (5) and (6) may be expressed as a surface gradient of a potential function

$$\underset{\sim}{w} = \nabla^{(2)}\nu \equiv \nabla\nu - \nabla V \, \frac{\nabla V\cdot\nabla\nu}{|\nabla V|^2} \ . \tag{7}$$

(A surface gradient is a gradient with the normal component subtracted off.) Hence, with only the assumption that flux surfaces exist, we may write any divergence-free field $\underset{\sim}{B}$ in terms of a scalar potential ν

$$\underset{\sim}{B} = \nabla V \times \nabla^{(2)} \nu \ . \tag{8}$$

Such characterizations of the magnetic field for equilibrium plasmas are discussed by Grad and Rubin[5] and by Kruskal and Kulsrud[6]. Although we are not discussing equilibria we shall often refer to their calculations when the details follow in the same way. The potential ν is not single valued on the torus; in both of the above references, the period of ν is calculated to be

$$\begin{aligned} \oint_{C_1} d\underset{\sim}{\ell} \cdot \nabla^{(2)} \nu &= \dot{\psi}_2(V) \\ \oint_{C_2} d\underset{\sim}{\ell} \cdot \nabla^{(2)} \nu &= -\dot{\psi}_1(V) \end{aligned} \tag{9}$$

where

$$\dot{\psi}_j(V) \equiv d\psi_j/dV \ .$$

These periods may be used to write an explicit representation for the potential and for the magnetic field

$$\begin{aligned} \nu &= \theta_1 \dot{\psi}_2 - \theta_2 \dot{\psi}_1 + \lambda(\theta_1, \theta_2, V) \\ \underset{\sim}{B} &= \dot{\psi}_2 \ \nabla V \times \nabla^{(2)} \theta_1 - \dot{\psi}_1 \ \nabla V \times \nabla^{(2)} \theta_2 + \nabla V \times \nabla^{(2)} \lambda \ . \end{aligned} \tag{10}$$

Note that λ is a single valued function on the torus and that it depends upon the choice of coordinates (we shall see later that it can be made constant on each surface for the appropriate choice of coordinates). For fields associated with a given pair of fluxes,

λ characterizes the distribution of the field over each flux surface.

B. Magnetic Energy (Hodges' Theorem)

Up to now we have considered the class of all possible magnetic fields associated with a given set of flux surfaces and with a given pair of fluxes (ψ_1, ψ_2) for each surface. We now consider only that magnetic field configuration from the above class which minimizes the magnetic energy $\int d^3x\ \frac{1}{2}B^2$. If we use the representation of the magnetic field given by equation (10) we see that the single-valued function λ is still free when the functions $V(x), \psi_1(V)$ and $\psi_2(V)$ are specified and when any specific choice is made for the coordinates θ_1 and θ_2. If we extremize the magnetic energy by varying λ (see Eqs. (H6) and (H7) in Kruskal and Kulsrud[6] or Equation (25) in Grad and Rubin[5]) we find

$$\nabla V \cdot \nabla \times \underset{\sim}{B} = 0 \ . \tag{11}$$

As described in the last section, any vector field which satisfies equations of the form (11) and (4) may be written as the surface gradient [Eq. (7)] of a potential function

$$\underset{\sim}{B} = \nabla^{(2)} \Phi \ . \tag{12}$$

The potential Φ is not single valued on the torus; according to Ampere's law [Eq. (2)], the period of Φ is the total current passing through the contour C_i

$$\oint_{C_i} d\underset{\sim}{\ell}\cdot\nabla^{(2)}\phi = I_i \tag{13}$$

Since contour integrals over a surface gradient are the same for any choice of homologous contours, Eq. (13) implies that current and flux surfaces coincide and that the total currents are surface quantities $I_i = I_i(V)$.

From the second variation of magnetic energy with respect to λ it follows that the extremum is always a local minimum of the magnetic energy on each surface. All higher order variations are zero. This suggests that the extremum is an absolute minimum. This property can be shown explicitly by using the special coordinate system to be defined by Eqs. (21) and (23). It will be shown that the minimum is attained when λ is a surface quantity and therefore drops out of the representation for $\underset{\sim}{B}$.

The magnetic energy, so minimized, may now be written completely in terms of the total currents and the differential fluxes. This application of Hodges' theorem^2 may be demonstrated explicitly^{5,6} by using Eq. (10) and (11) to write

$$B^2 = \nabla V\cdot\nabla\times\{[\theta_1\dot{\psi}_2 - \theta_2\dot{\psi}_1 + \lambda]\underset{\sim}{B}\} \ .$$

If the flux surfaces are labeled so that $V = 0$ is on the magnetic axis and $V = C$ is the volume of a rigid, perfectly conducting, toroidal outer boundary, the integration over volume may be written

$$\int d^3x \ldots = \int_0^C dV \int_{S_V} ds \, \frac{1}{|\nabla V|} \ldots \ . \tag{14}$$

We then use Stokes' theorem and the fact that $\nabla V/|\nabla V|$ is the unit normal to each flux surface to write

$$M \equiv \int_{\min} d^3x \;\; \tfrac{1}{2}B^2 = \tfrac{1}{2}\sum_{i=1}^{2} \int_o^c dV \;\; I_i(V)\dot{\psi}_i(V) \tag{15}$$

Note that the single-valued function λ does not contribute; the minimum magnetic energy M depends upon only surface quantities. Eq. (15) is true only when flux and current surfaces coincide--only when no current ($\underset{\sim}{j} \equiv \nabla \times \underset{\sim}{B}$) passes through flux surfaces.

C. Specific Inductance

For any system with simply nested flux surfaces and coincident current surfaces we now assert that there is a linear relation between the total currents and the differential fluxes

$$I_i(V) = \sum_{j=1}^{2} \Lambda_{ij}(V)\dot{\psi}_j(V) \tag{16}$$

where $\Lambda_{ij}(V)$ depends upon only the geometry of the flux surfaces (is independent of I_i and $\dot{\psi}_j$). We shall refer to Λ_{ij} as the specific inductance matrix. If we insert this linear relation into Eq. (15) we derive an expression for the minimized magnetic energy in which the geometry and flux profiles are separately factored

$$\int_{\min} d^3x \; \tfrac{1}{2}B^2 = \tfrac{1}{2}\sum_{i,j=1}^{2} \int_o^C dV \;\; \dot{\psi}_i(V)\Lambda_{ij}(V)\dot{\psi}_j(V) \; . \tag{17}$$

This formulation is due to Harold Grad.[1]

For vacuum fields, the specific inductance matrix is the reciprocal of the derivative with respect to volume of the ordinary inductance matrix. (Λ_{ij} should rigorously be called the specific susceptance matrix and its reciprocal called the inductance. Only the more well-known term inductance will be used here. No confusion arises because there is no need to invert the matrix.) The specific inductance, relating the total currents to the fluxes between differentially close flux surfaces, can be applied to systems with distributed current densities (provided the current density lies in the flux surfaces), while the ordinary inductance, relating total currents to total fluxes, is properly applied only to vacuum magnetic fields.

We shall derive three representations for $\Lambda_{ij}(V)$ and prove that it is a positive definite, symmetric matrix which depends upon only the geometry of the flux surfaces.

The most elegant derivation of the specific inductance, due to Grad[1], relies on a uniqueness theorem for the potentials. According to Eqs. (8) and (12), the magnetic field is characterized by two potential functions ν and Φ which are the solutions to the following elliptic system of equations

$$\nabla V \times \nabla^{(2)}\nu = \nabla^{(2)}\Phi \ . \qquad (18)$$

By the nature of the surface gradient [Eq. (7)], Eq. (18) represents only two equations (one for each component lying in the flux surface) for the two unknowns ν and Φ. We use the lemma[2,5] that the solutions of Eq. (18) are uniquely determined on a flux surface given only the two periods of Φ, which are the total currents I_i. Now Eq. (18) may be rearranged by crossing it with ∇V

$$- \frac{\nabla V}{|\nabla V|^2} \times \nabla^{(2)}\phi = \nabla^{(2)}\nu \; . \tag{19}$$

By the same lemma, the solutions to Eq. (19) are uniquely determined on a flux surface (provided $|\nabla V| \neq 0$) given only the periods of ν, which are the differential fluxes $d\psi_j/dV$. Since the same solutions to this set of linear equations are uniquely determined given either set of periods, the periods must be linearly dependent, as in Eq. (16). The coefficients Λ_{ij} in this linear relation between the periods must depend upon only the common given information, which is the geometry of the flux surfaces.

In order to compute the specific inductance matrix Λ_{ij} we may use any solutions ν and ϕ of Eq. (18), compute their periods, and insert into Eq. (16). If the off-diagonal terms are present in Λ_{ij}, two sets of solutions for ν and ϕ with distinct periods are needed in order to invert the matrix Eq. (16).

An equivalent but more intuitively appealing derivation follows if we insert the characterization of the magnetic field given by Eq. (10) into the condition that the current density lies in the flux surfaces [Eq. (11)]. This gives an elliptic equation for the single valued function $\lambda(\theta_1, \theta_2, V)$

$$\begin{aligned} &\sum_{ij} \nabla V \times \nabla\theta_i \cdot \nabla V \times \nabla\theta_j \; \lambda_{\theta_i\theta_j} \\ &+ \sum_j \nabla V \cdot \nabla \times (\nabla V \times \nabla\theta_j)[(-1)^{j+1}\dot{\psi}_{j+1}(V) + \lambda_{\theta_j}] = 0 \end{aligned} \tag{20}$$

Suppose we choose a coordinate system such that

$$\nabla V \cdot \nabla \times (\nabla V \times \nabla \theta_j) = 0 \tag{21}$$

We are then left with only the second order terms in the equation for λ

$$\sum_{ij} \nabla V \times \nabla \theta_i \cdot \nabla V \times \nabla \theta_j \lambda_{\theta_i \theta_j} = 0 \ .$$

For the solution to be a single-valued function on each toroidal flux surface it must be doubly periodic in (θ_1, θ_2). By the maximal principle,[10] the solution cannot assume either a local maximum or minimum within any boundary contours prescribed on the surface. Since any closed contours on the torus may be used for the boundary, it follows that λ must be constant over the entire flux surface and therefore does not contribute to $\underset{\sim}{B}$ in Eq. (10). Now if we use Eq. (10) and Eq. (3) to compute the total current I_i we find

$$\Lambda_{ij}(V) = (-1)^j \oint_{C_i} d\underset{\sim}{\ell} \cdot \nabla V \times \nabla \theta_{j+1} \tag{22}$$

When using Eq. (22) to compute the specific inductance matrix, the coordinates θ_1 and θ_2 must satisfy Eq. (21), a purely geometric condition, and they must have unit periods

$$\oint_{C_i} d\underset{\sim}{\ell} \cdot \nabla \theta_j = \delta_{ij} \ . \tag{23}$$

This is a constructive derivation which explicitly shows that the specific inductance is a purely geometric entity.

Note that Eq. (21) implies that the vector field $\nabla V \times \nabla\theta_j$ must be a surface gradient of some potential

$$\nabla V \times \nabla^{(2)}\theta_j = \nabla^{(2)}\phi_j \ .$$

Hence, the coordinates θ_1 and θ_2 may be thought of as the particular choices of the magnetic potential ν with normalizations given by Eq. (23). The lemma, referred to before, that solutions to equations of this kind are determined by their periods uniquely up to a constant on each surface can be proved by the methods used here. Suppose distinct solutions with the same periods existed. They would differ by a single valued function which would satisfy the homogeneous part of Eq. (20). The maximal principle[10] is then be used to show that this function is constant on each surface.

It should be pointed out[8] that the coordinate system just derived ((V,θ_1,θ_2) satisfying Eqs. (21) and (23)) provides an alternative to Hamada coordinates.[9] Hamada coordinates exist only in equilibrium -- when $\underset{\sim}{j} \times \underset{\sim}{B} = \nabla p$ for $\underset{\sim}{j} \equiv \nabla \times \underset{\sim}{B}$ and $p(V)$ a surface quantity. Our coordinate system exists in or out of equilibrium provided the magnetic energy is minimized within each surface. Even in equilibrium, our coordinate system is equivalent to Hamada coordinates if and only if[8]

$$\nabla V \cdot \nabla \times \underset{\sim}{j} = 0 \ . \tag{24}$$

In general, Hamada coordinates are functionals of $\dot{\psi}_i$ (i = 1 or 2) while our coordinate system depends upon only the shape of the flux surfaces. The field lines of both $\underset{\sim}{B}$ and $\underset{\sim}{j}$ are straight in Hamada coordinates. Our coordinate system is one of the possible coordinates in which only the $\underset{\sim}{B}$ lines are straight.

If the metric elements $g_{ij} \equiv \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j$, $\underset{\sim}{e}_j \equiv (-1)^j \nabla V \times \nabla\theta^{j+1}/D$, $D \equiv \nabla\theta^1 \times \nabla\theta^2 \cdot \nabla V$, are known for the coordinates satisfying Eqs. (21) and (23), it is useful to use a representation for Λ_{ij} derived by Günther Spies.[8] Covariant and contravariant components are used to express the square of the $\underset{\sim}{B}$ field in the magnetic energy Eq. (17)

$$B^2 = B_i B^i, \quad B^i = \dot{\psi}^i D, \quad B_i = g_{ij} B^j .$$

After using Eq. (14), we may identify the specific inductance factor in Eq. (17) with the expression

$$\Lambda_{ij} = \oint_{S_V} \frac{dS}{|\nabla V|} g_{ij} D^2 = \int\int d\theta^1 d\theta^2 \, g_{ij} D . \tag{25}$$

From the well known properties of the metric tensor, it follows that Λ_{ij} is symmetric and positive definite. Note that the equation for the coordinates [Eq. (21)] implies

$$(g_{1j}D)_{\theta^2} = (g_{2j}D)_{\theta^1}$$

so that only one integration is needed in Eq. (25).

III. Formal Variation of Potential Energy

A. Constraints

We shall now use a simple, scalar pressure, magnetohydrodynamic model to investigate the equilibrium and stability of toroidal plasmas with distributed profiles. Details of the ideal magnetohydrodynamic model are given by Bernstein, Frieman, Kruskal, and Kulsrud.[7] The total energy

$$E = \int d^3x \, \{\tfrac{1}{2}\rho v^2 + \tfrac{1}{2}B^2 + \frac{p}{\gamma-1}\}$$

is a constant of the equations of motion assuming that fluid elements expand or compress adiabatically during any motion and assuming (here) that there are no electric fields or body forces. We define the state of the system to be those properties (B and p) which are independent of the velocity v. In the energy variational principle, we wish to compare the potential energy

$$W \equiv \int d^3x \, \{\tfrac{1}{2}B^2 + \frac{p}{\gamma-1}\} \qquad (26)$$

of only those states which are accessible to each other by a possible motion of the plasma. The condition of accessibility is usually built in to the variation of W by using "constraints"--that is conditions independent of velocity which are to be the same in the perturbed and unperturbed states. These will be enumerated below. It has been shown[5,6] that, subject to these constraints, $\delta W = 0$ is equivalent to other definitions of static equilibrium. We define the system to be unstable if and only if any other accessible state has a minimized potential energy lower than that of the equilibrium being considered.

We have already imposed two constraints on all states of the system to be considered (to apply to both the equilibrium and perturbed states):

(1) Nested flux surfaces exist; it follows that $\underset{\sim}{B}\cdot\nabla V = 0$:

(2) The magnetic energy is minimized within each surface; it follows that $\underset{\sim}{j}\cdot\nabla V = 0$.

In order to minimize the rest of the potential energy we now demand that

(3) The thermodynamic energy is minimized within each surface.[3] This implies that the plasma is to be in thermodynamic equilibrium within each surface; the pressure, density, temperature, entropy, etc. are all uniform over each surface. However, the plasma generally need not be in equilibrium perpendicular to the flux surfaces. In particular, we see from the ideal magnetohydrodynamic equation of motion

$$\rho \, d\underset{\sim}{v}/dt = -\nabla p + \underset{\sim}{j} \times \underset{\sim}{B}$$

that constraints (2) and (3) imply that there is a component of the inertial term $\rho \, d\underset{\sim}{v}/dt$ only perpendicular to each flux surface.

After satisfying these three constraints, the potential energy (26) assumes the form

$$\Phi \equiv \int_0^C dV\{\frac{1}{2}\,\dot{\psi}_i(V)\Lambda_{ij}(V)\,\dot{\psi}_j(V) + \frac{p(V)}{\gamma-1}\} \quad . \tag{27}$$

where the summation convention is used over repeated indicies $i,j = 1,2$.

Three more constraints are needed to assure that the perturbed states are accessible to the equilibrium state. These are:

(4) The topology of each field line within the perfectly conducting plasma is preserved during a motion of the fluid. Since flux is carried by the plasma, it follows that one flux must be a given function of the other during a variation

$$\psi_2 = \psi_2(\psi_1) \tag{28}$$

so that the rotational transform

$$\iota\,(\psi_1) = 2\pi\, d\psi_2/d\psi_1$$

is preserved within each plasma shell. We shall take the rotational transform as part of the data which describes the equilibrium state. Eq. (28) implies that we need vary only one flux, $\psi \equiv \psi_1(V)$.

(5) Entropy is conserved during the motion of each fluid shell; that is, the pressure and density vary according to the adiabatic law.

(6) Fluid mass is conserved during the motion of each fluid shell.

The last two constraints lead to a relation between the variation of pressure and the variation of flux. These constraints are most easily expressed by using a "convective variation" which follows the fluid shells rather than the ordinary variation taken at constant volume.

Consider a variation of the flux profile as shown in Fig. (2). A given set of flux surfaces may be labeled by either V or ψ. Let δ refer to the variation with V as the independent parameter:

$$\delta \equiv \delta\psi \frac{\partial}{\partial\psi} \tag{29}$$

Let Δ refer to the convective variation which follows a given fluid element, where ψ is the independent parameter:

$$\Delta \equiv \Delta V \frac{\partial}{\partial V} \tag{30}$$

Note that $\delta V = 0$ and $\Delta\psi = 0$. We can graphically determine the inverse function relation from Fig. (2):

$$\delta\psi = - \frac{d\psi}{dV} \Delta V \tag{31}$$

Then, for any function of V and ψ we may use the chain rule

$$\frac{d}{dV} = \frac{d\psi}{dV} \frac{\partial}{\partial\psi} + \frac{\partial}{\partial V}$$

together with Eqs. (29), (30), and (31) to derive the identity relating the two kinds of variation

$$\Delta = \delta - \frac{\delta\psi}{\dot{\psi}(V)} \frac{d}{dV} \qquad (32)$$

The shell of plasma between two flux surfaces is labeled by the fluxes of the bounding surfaces. As the flux profile is varied, the plasma moves with the flux so that each plasma shell retains the same flux labeling but may change its volume. After each plasma shell comes into thermodynamic equilibrium, we may express the adiabatic compression or expansion of the plasma by

$$\frac{\Delta p}{p} = \gamma \frac{\Delta \rho}{\rho} \qquad (33)$$

where ρ is the fluid density and γ is the ratio of specific heats (from kinetic theory, $\gamma = 1+2/(\text{number of degrees of freedom})$). In order to calculate $\Delta\rho$ we need to use the conservation of mass within each fluid shell, constraint (6),

$$\Delta(\rho \; dV) = 0 \; . \qquad (34)$$

Note that the operators Δ and $\underline{d}$ commute. We may now use the last three equations to write

$$\delta p = (1-\gamma) \frac{dp}{dV} \frac{\delta\psi}{\dot{\psi}} + \frac{d}{dV}\left(\gamma p \frac{\delta\psi}{\dot{\psi}}\right) \; . \qquad (35)$$

Note that it is unnecessary and misleading to assume that the entropy is constant over the plasma profile. We have assumed only that the entropy and mass of each fluid shell remains constant under the variation.

We may vary the flux profile and the inductance either separately or both simultaneously. In general, we must also vary the total current profile $I_i(V)$ in order to always satisfy Eq. (16). However, the variation of $I_i(V)$ never appears explicitly. The variation of flux and inductance separately will give not only all the possible equilibrium conditions but also a necessary and sufficient stability condition. That is, the cross term in the second variation with both flux and inductance varied simultaneously and independently does not affect the stability result.

B. Variation of Profile with Fixed Inductance

Conceptually the simplest variation of the minimum potential energy (27) is to vary the flux as a function of volume while holding the inductance as a function of volume fixed. That is, we rearrange the flux labeling of the surfaces without changing their shape. We take the flux to be fixed at the magnetic axis, $\delta\psi(0) = 0$, and at the perfectly conducting wall, $\delta\psi(C) = 0$. Depending upon which flux is used, one condition is true by definition and the other condition may be relaxed by using a free boundary variation.

Using the fact that Λ_{ij} is a symmetric matrix, we have for the first variation of potential energy (2)

$$\delta\Phi = +\int_0^C dV\{\psi_{iV}\Lambda_{ij}(\delta\psi_j)_V + \delta p/(\gamma-1)\} .$$

Using the constraint that one flux is a function of the other,

$$\delta\psi_i = \psi_{iV}\,\delta\psi/\psi_V\ ,$$

and using Eq. (35) to relate the variation of pressure to variation of flux, and integrating by parts with fixed end conditons, we find that γ drops out of the first variation

$$\delta\Phi = -\int_o^C dV\{\psi_{iV}(\Lambda_{ij}\psi_{jV})_V + dp/dV\}\,\delta\psi/\psi_V\ .$$

Our first equilibrium equation results from the extremum condition $\delta\Phi = 0$ for arbitrary flux variations

$$\psi_{iV}(\Lambda_{ij}\ \psi_{jV})_V + dp/dV = 0 \tag{36}$$

where

$$\psi_{iV} \equiv d\,\psi_i\,(V)/dV\ .$$

The second variation is found by using $\delta^2\psi = 0$

$$\delta^2\psi_i = \ddot{\psi}_i(\delta\psi/\dot{\psi})^2 - \dot{\psi}_i\ \delta\psi\delta\dot{\psi}/\dot{\psi}^2$$

$$\delta(\frac{dp}{dV}/\dot{\psi}) = \frac{d}{dV}[\gamma p\ \frac{d}{dV}(\delta\psi/\dot{\psi})]/\dot{\psi} + \frac{d}{dV}[\dot{p}(V)/\dot{\psi}]\delta\psi/\dot{\psi}^2\ .$$

After some algebra we find

$$\delta^2\Phi = \int_o^C dV\{[\psi_{iV}\Lambda_{ij}\ \psi_{jV} + \gamma p]\ \delta\phi_V{}^2 + \psi_{iV}(\Lambda_{ijV}\psi_{jV})_V\ \delta\phi^2\} \tag{37}$$

where

$$\delta\phi \equiv \delta\psi/\dot{\psi} \quad (\text{assuming } \dot{\psi}(V) \neq 0)$$

is an arbitrary test function with $\delta\phi(0) = \delta\phi(C) = 0$. Note that the expression in square brackets is positive definite; hence there can be negative contributions from only the last term in Eq. (37). The final minimization for this part of the potential energy is a minimization with respect to the test function $\delta\phi$ which, after normalizing $\delta\phi$, leads to an ordinary Sturm-Liouville equation for $\delta\phi$. The sign of the lowest eigenvalue determines the stability of these fixed geometry modes.

In the special case of straight, coaxial circular cylinders (screw pinch) of length L with periodic end conditions we have the following inductance matrix

$$\Lambda_{11} = L^2 \qquad \Lambda_{22} = \frac{4\pi}{L} V$$

$$\Lambda_{12} = \Lambda_{21} = 0 \ .$$

If the current is purely longitudinal and concentrated near the axis, it is possible to find unstable profile distributions of the form

$$\dot{\psi}_2(V) \sim [aV/C - 1]^{-1} \quad \text{for } a \gg 1 \ .$$

The instability is a breathing mode localized near the axis. This and other examples will be discussed in future papers.

C. Incompressible Variations

For the second subclass of variations we imagine that the flux surfaces change their shape but that each plasma shell preserves its volume by moving to a new surface. Hence we take $\delta\psi = 0$ and therefore $\delta p = 0$. We call these "incompressible variations" because the fluid shells are neither compressed or expanded. However, as the shells change shape, individual fluid elements may compress or expand during their motion within the shell as they reach surface equilibrium (constraint (3)). Hence these "incompressible variations" should not be confused with variations of an incompressible fluid. The latter are only a subclass of the former.

We label the flux surfaces by their volume and parameterize their shapes by a countable number of functions $a_n(V)$. The functions $a_n(V)$ might for example, be the ellipticity of elliptical surfaces, or the relative displacements of their centers, or the amplitude or the phase of a helical distortion, etc. Since the specific inductance is derived from first order differential equations we conclude that it is a function of only the zero and first derivatives of $a_n(V)$

$$\Lambda_{ij} = \Lambda_{ij}(V;\ a_n(V),\ a_n'(V);\ n = 1,2,\ldots)$$

where $a_n'(V) \equiv d\,a_n(V)/dV$; no derivatives higher than first order appear.

We vary each function $a_n(V)$ independently. It follows that there are as many equilibrium conditions as there are such functions; each has the form

$$\psi_{iV}\Lambda_{ija_n}\psi_{jV} - (\psi_{iV}\Lambda_{ija'_n}\psi_{jV})_V = 0 \qquad (38)$$

with

$$\psi_{iV}\, \partial\Lambda_{ij}/\partial a_n'(V)\, \psi_{jV}\, \delta a_n(V)\, \Big|_{V=0} = 0\ . \qquad (39)$$

Note that $\delta a_n = 0$ at $V = C$ because the wall is assumed to be rigid and perfectly conducting. Since the total volume of the system remains unchanged, it is always possible to take $\delta\psi = 0$ and $\delta p = 0$ at $V = 0$ and at $V = C$.

The equilibrium conditions given by Eq. (38) may be thought of as separate extrema for each one parameter family of surfaces being considered. If the class of surfaces is limited -- for example, to elliptical surfaces with variable ellipticity -- Eq. (38) represents a constrained equilibrium -- as opposed to a true equilibrium where the class of surfaces is limited only by smoothness conditions. The concept of constrained equilibria is a useful tool for approximation. But, in order to improve such approximations, we need to characterize the class of surfaces to be considered by a complete set of parameters. The rest of this work is independent of Eq. (38); all other results may be used if the equilibrium is known from an independent calculation.

For incompressible variations of the fluid shells, the stability with respect to infinitesimal distortions of the flux surfaces can be determined from the simple form

$$\delta^2\Phi = \int_0^C dV\, \tfrac{1}{2}\, \psi_{iV}\, \delta^2\Lambda_{ij}\, \psi_{jV}\ . \qquad (40)$$

It has been verified that this result agrees with the classical result for a screw pinch where an arbitrary infinitesimal distortion of the flux surfaces can be decomposed into a spectrum of helical distortions with $k \neq 0$ or $m \neq 0$.

We may consider stability with respect to finite distortions of the flux surfaces provided $\psi(V)$ is the same in the final state as in the equilibrium state. That is, we replace δ^2 in Eq. (40) by a symbol for the finite difference. We are therefore able to study nonlinear stability with respect to incompressible variations. We are not able to find a simple expression for the finite variation of the flux profile with fixed inductance because the conversion between convective and ordinary variation relied on the infinitesimal nature of the variations.

D. General Variation

In general, the variations $\delta\psi$ and $\delta\Lambda$ are independent of one another if the total currents I_i are not fixed functions of volume. Eqs. (36) and (38) still represent all the equilibrium conditions subject to our constraints. However, cross terms are possible in the second variation of the potential energy

$$\begin{aligned}\delta^2\Phi = \int_0^C dV\{[\psi_{iV}\Lambda_{ij}\,\psi_{jV} + \gamma p]\delta\phi_V{}^2 + \psi_{iV}(\Lambda_{ijV}\psi_{jV})_V\,\delta\phi^2 \\ + \psi_{iV}\,\delta\Lambda_{ij}(\psi_{jV}\,\delta\phi)_V + \tfrac{1}{2}\,\psi_{iV}\,\delta^2\Lambda_{ij}\,\psi_{jV}\}\end{aligned} \qquad (41)$$

where

$$\delta\phi \equiv \delta\psi/\psi_V, \quad \psi(V) \equiv \psi_1(V) .$$

If either the fixed inductance variation or the incompressible variation were unstable we would need to go no further than to consider the special second variations given by Eqs. (37) or (40). However, if both special variations were positive, we would need to examine the effect of the cross term in Eq. (41) in order to find a sufficient as well as necessary condition for stability.

E. Equivalence with the Classical Energy Principle

We shall show that the energy principle that has been derived here, using constraints (1) through (6), is equivalent to the partially minimized ordinary energy principle[7] when used for the purpose of distinguishing stability from instability. However, growth or decay rates cannot be calculated from our variational principle directly.

Bernstein, Frieman , Kruskal and Kulsrud[7] point out in section 4b that it is sufficient to normalize only that component of the displacement $\underset{\sim}{\xi}$ which is orthogonal to the p_o = constant surfaces in order to distinguish stability from instability. The condition δW be minimum yields an Euler equation with three components. With only the orthogonal component of $\underset{\sim}{\xi}$ normalized, the two components of the Euler equation in the surface are, to first order in $\underset{\sim}{\xi}$,

$$\underset{\sim}{j}\cdot\nabla p = 0 \quad \text{and} \quad \underset{\sim}{B}\cdot\nabla p = 0 \; .$$

If we assume the existence of well defined flux surfaces (constraint (1)) it follows from these equations that p is constant on flux surfaces (constraint (3)) and that $\underset{\sim}{j}\cdot\nabla V = 0$ (constraint (2)). Hence, for infinitesimal perturbations we have equivalence of the energy principles in the sense that

$$\min \delta W \gtrless 0 \leftrightarrows \delta^2\Phi \gtrless 0 \; .$$

IV Special Calculations of Specific Inductance

It seems to be possible to find exact formulas, in terms of one integration, for the specific inductance of any nested set of smooth flux surfaces with an ignorable coordinate. We shall derive such formulas for a straight cylinder, an axisymmetric torus, and a straight helically symmetric system, each with an arbitrary cross section. For the first two cases we need to use only Ampere's law, without appealing to the general methods derived in section II. For the case of helical symmetry we use a method suggested by Grad. That is, we find two simple solutions for the two potentials of Eq. (18) and then determine their periods.

In the case of a straight cylinder and an axisymmetric torus, it is possible to perform the integration analytically when the cross sections are circles with relative displacements between their centers. By collecting these exact solutions we may begin to develop intuition about the specific inductance in general.

A. Straight Cylinder

We consider straight cylindrical flux surfaces with arbitrary, smooth cross sections given by $V = V(x,y)$. If the only component of current is parallel to the ignorable z coordinate then the vector potential has only a z component $\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \nabla A(x,y) \times \hat{z}$ and all flux is aximuthal. If we set $A = 0$ along a reference line parallel to the z axis, we can use Stoke's theorem on the definition of the flux [Eq. (1)] to show

$$\Psi(x,y) = A(x,y)\, L$$

where L is the periodic length of the cylinder. This flux is a function of V alone

$$\Psi(x,y) = \psi_1(V(x,y)) \ .$$

These equations are used in Ampere's law Eq. (3) to find the relation between longitudinal current and azimuthal flux

$$I_2 = \oint_{C_2} d\underset{\sim}{\ell}\cdot\underset{\sim}{B} = \oint_{C_2} d\underset{\sim}{\ell}\cdot\frac{1}{L}\,\nabla\Psi \times \hat{z}$$

$$= \frac{1}{L}\oint_{C_2} d\underset{\sim}{\ell}\cdot\nabla V(x,y) \times \hat{z}\;\; \psi_1(V) \;.$$

Hence, the azimuthal specific inductance is

$$\Lambda_{22}(V) = \frac{1}{L}\oint_{C_2} d\underset{\sim}{\ell}\cdot\nabla V(x,y) \times \hat{z} \tag{42}$$

where C_2 is any closed contour on the flux surface wrapping once around the cylinder.

For the calculation of the longitudinal specific inductance, Λ_{11} we place contour C_1 along the ignorable z coordinate with the return element along a reference line. From Ampere's law we find a relation between the azimuthal current and the longitudinal field

$$I_{az} = B_{long}\, L$$

Since I_{az} is independent of the choice of C_1 contour taken on any given flux surface, we conclude that B_{long} is uniform over each flux surface, regardless of the cross sectional shape. If dS_1 is the area of the cut surface S_1 between differentially close flux surfaces, we use $dV = L\, dS_1$ to show

$$\frac{d\psi_2}{dV} = B_{long}\frac{dS_1}{dV} = B_{long}/L = I_{az}/L^2$$

Hence, the longitudinal specific inductance

$$\Lambda_{11} = L^2 \tag{43}$$

is independent of the cross sectional shape of the flux surfaces. Note that there is no mutual inductance for straight cylinders.

One of the simplest non-trivial applications of Eq. (42) is to the case of non-concentric circular cylinders. If r is the radius of such a surface and a(r) is the displacement of its center along the x axis, then these surfaces may be represented by

$$V/\pi L = r^2 = y^2 + (x - a(r))^2 .$$

This form may be used in Eq. (1) to find

$$\Lambda_{22} = \frac{4\pi}{L} \frac{V}{\sqrt{1 - (a'(r))^2}} . \tag{44}$$

If a(r) = 0, we have the specific azimuthal inductance for concentric circular cylinders

$$\Lambda_{22} = \frac{4\pi}{L} V \tag{45}$$

This may be derived from the ordinary vacuum field inductance, which is proportional to ln(r), by first applying d/dV and then taking the reciprocal. Returning to the nonconcentric result (3), we note that $\Lambda_{22} \rightarrow \infty$ as $|da/dr| \rightarrow 1$. Geometrical analysis indicates that differentially close flux surfaces touch and cross each other when $|da/dr| \geq 1$. Intuitively, then, the specific inductance behaves like the capacitance between close surfaces and it becomes infinite as they touch.

B. Axisymmetric Torus

If we use cylindrical coordinates (R,ϕ,z) and take ϕ as the ignorable coordinate, a calculation similar to the above yields the specific inductance for axisymmetric toroidal flux surfaces with arbitrary cross sections and relative displacements

$$\Lambda_{22} = \oint_{C_2} d\underset{\sim}{\ell}\cdot\nabla V \times \hat{\phi}/(2\pi R) \tag{46}$$

$$\Lambda_{11} = 2\pi dV/\int_{S_1 V}^{V+dV} \frac{ds}{R} \ . \tag{47}$$

Again the matrix is diagonal.

As a special case, consider symmetric toruses with circular cross sections[4] of radius r with their centers displaced in the plane of the torus by $a(r)$ relative to the magnetic axis at $R = R_o$; positive values of $a(r)$ represent a displacement inward from the magnetic axis toward the center line. Such toruses may be represented by the equation

$$V = 2\pi^2(R_o-a)r^2 = 2\pi^2(R_o-a)[z^2+(R-R_o+a)^2] \ .$$

If we let $D \equiv da/dr$ and $\varepsilon \equiv r/R$ (the aspect ratio), the exact results are

$$\Lambda_{22} = \frac{2V/R}{1+D/\varepsilon+\varepsilon D/2}\left\{\frac{1}{\sqrt{1-\varepsilon^2}} + \frac{D/\varepsilon}{\sqrt{(1+\varepsilon D/2)^2-D^2}}\right\} \tag{48}$$

$$\Lambda_{11} = 4\pi^2 R^2 / \{ \frac{1+D/\varepsilon+\varepsilon D/2}{\sqrt{1-\varepsilon^2}} - D/\varepsilon \} \qquad (49)$$

where $(1+\varepsilon D/2)^2 > D^2$ is the condition that no surfaces touch each other. Note that the inductances are finite at $1+D/\varepsilon+\varepsilon D/2 = 0$.

C. Helical Symmetry

For systems topologically equivalent to a periodic cylinder, the computations are most easily carried out if we retain the unit vectors of cylindrical coordinates $(\hat{z}, \hat{\theta}, \hat{r})$ but change variables to (z, θ, R) where $R = R(z, \theta, V)$. The $\hat{z}$ and $\hat{\theta}$ components of Eq. (18) are

$$\begin{aligned} -\frac{R}{R_V} Q\, \nu_\theta &= (R^2+R_\theta^2)\phi_z - R_\theta R_z \phi_\theta \\ \frac{R}{R_V} Q\, \nu_z &= -R_\theta R_z \phi_z + (1+R_z^2)\phi_\theta \end{aligned} \qquad (50)$$

where

$$Q \equiv 1 + R_z^2 + R_\theta^2/R^2 .$$

The $\hat{r}$ component of Eq. (18) is a linear combination of these equations. With these conventions, the periods are

$$\begin{aligned} I_1 &= \int_0^L dz\ \phi_z, \quad I_2 = \int_0^{2\pi} d\theta\ \phi_\theta , \\ \mathring{\psi}_1 &= -\int_0^{2\pi} d\theta\ \nu_\theta, \quad \mathring{\psi}_2 = \int_0^L dz\ \nu_z . \end{aligned} \qquad (51)$$

Any surface with helical symmetry may be represented by $R = R(\xi, V)$ where $\xi \equiv m\theta + kz$. If the surfaces were straight circular cylinders, where $V = \pi R^2 L$, the solutions to Eq. (50) would be

$$\nu = C_1 z + C_2 \theta$$

$$\phi = +2VC_1\theta - 2\pi L C_2 z \ .$$

A solution is found for the helical system by adding $\tilde{\nu}(\xi, V)$ and $\tilde{\Phi}(\xi, V)$ to these solutions. From Eq. (50), these new functions satisfy the ordinary differential equations

$$(m^2 + k^2R^2)\,\frac{d\tilde{\nu}}{d\xi} = k[2VRR_V - R^2]C_1 + m[2\pi LRR_V - 1]C_2$$

$$(m^2 + k^2R^2)\,\frac{d\tilde{\Phi}}{d\xi} = m[2V - RQ/R_V]C_1 + k[RQ/R_V - 2\pi LR^2]C_2 \ .$$

The results of inserting the inegrals of these equations [Eq. (51)] into Eq. (16) and solving for Λ_{ij} are

$$\Lambda_{11} = L^2\{1 + \beta P + \alpha(M+N) + \alpha m^2(M+N)^2/(1 - \alpha N - m^2 M)\}$$

$$\Lambda_{22} = \frac{4\pi V}{L}\{1 + m^2 P + \alpha N + \alpha\beta N^2/(1 - \alpha N - m^2 M)\} \qquad (52)$$

$$\Lambda_{12} = \Lambda_{21} = 2Vmk\{P - \alpha N(N+M)/(1 - \alpha N - m^2 M)\} \ .$$

Here we have represented the surfaces by

$$\pi R^2 L = (1 + \tilde{a}(\xi, V))V$$

and we have used the notations

$$M \equiv - \left\langle \frac{\tilde{a}}{\alpha+\beta\tilde{a}} \right\rangle, \quad N \equiv - \left\langle \frac{V\tilde{a}_V}{\alpha+\beta\tilde{a}} \right\rangle$$

$$P \equiv \left\langle \left(\frac{1}{4} \frac{a_\xi^{\,2}}{1+\tilde{a}} + \frac{V\tilde{a}_V (V\tilde{a})_V}{\alpha+\beta\tilde{a}} \right) / (1 + (V\tilde{a})_V) \right\rangle$$

$$\langle \ldots \rangle \equiv \int_0^{2\pi} \frac{d\xi}{2\pi} \ldots$$

$$\alpha \equiv m^2 + \beta$$

$$\beta \equiv Vk^2/\pi L \; .$$

No approximations have been used to derive these results from Eqs. (50), (51), and (16).

V. Conclusions

An essential step in this formulation of the energy principle is the minimization of potential energy before comparing states with a variational analysis. It is shown that a unique magnetic field minimizes the magnetic energy for a given set of nested flux surfaces and flux profiles. For this minimizing field, the specific magnetic energy can be factored into geometric and profile dependent matricies as in Eq. (17). The geometry-dependent matrix, called the specific inductance Λ_{ij}, is defined by Eq. (16). It can be represented by Eq. (22) or Eq. (25) if and only if we use the coordinate system defined by Eqs. (21) and (23). Alternatively, if any basis pair of total currents and differential fluxes is known for a given set of flux surfaces, Λ_{ij} may be computed directly from Eq. (16).

The equilibrium conditions given by Eqs. (36) and (38) result from extremizing the potential energy with respect to flux profile and surface shapes. It is necessary and sufficient for stability that the second variation given by Eq. (41) be positive. Eq. (37) represents the special case of a variation which leaves the surface shapes unchanged (fixed inductance). Eq. (40) represents a variation which leaves the flux profile unchanged (incompressible variation). The latter can be easily generalized to consider finite distortion of the flux surface shapes and hence nonlinear stability. It is shown in section III-E that the above stability criterion is equivalent to that of the classical energy principle.[7]

There are a number of promising applications for this new formulation. The one dimensional character of the integrations and the simple decomposition of the integrands makes this formulation especially suitable for computer studies of complicated geometries. The use of restricted families of surface shapes (circular, elliptical, etc.) for which there are simple calculations of the specific inductance is an aide to the analytic approximation of equilibria and the formulation of necessary conditions for stability. The coordinate system defined by Eqs. (21) and (23) provides an alternative to Hamada coordinates for the representation of stability criteria using a general perturbation analysis. Finally, the study of finite incompressible variations may be a useful approach to the study of nonlinear stability.

Acknowledgments

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Appendix

The following two derivations may be found in Grad and Rubin[5] and in Kruskal and Kulsrud.[6] They are presented here without any reference to equilibrium and with a redefinition of the coordinate conventions.

I. The Periods of ν

Given nested toroidal surfaces $V = V(\theta_1, \theta_2, r)$ and the conventions illustrated in Fig. 1 we employ Stokes' theorem to show

$$\oint_{C_1} d\ell \cdot \nabla^{(2)} \nu = \int_{\theta_2=1} d\ell \cdot \nabla^{(2)} \nu\theta_2 - \int_{\theta_2=0} d\ell \cdot \nabla^{(2)} \nu\theta_2$$

$$= -\oint_{C_V} d\ell \cdot \nabla^{(2)} \nu\theta_2 = \int_{S_V} dS \, \frac{\nabla V}{|\nabla V|} \cdot \nabla \times (\nabla^{(2)} \nu\theta_2) .$$

The integrand is transformed with vector identities, including

$$\nabla V \cdot \nabla \times \nabla^{(2)} \nu = 0$$

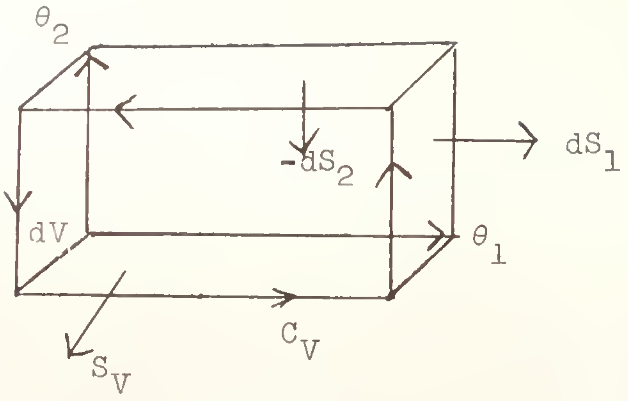
to show

$$\nabla V \cdot \nabla \times (\nabla^{(2)} \nu\theta_2) = \nabla \cdot (\theta_2 \nabla V \times \nabla^{(2)} \nu) .$$

We then use

$$\int_{S_V} \frac{dS}{|\nabla V|} \ldots = \frac{1}{dV} \int_V^{V+dV} d^3x \ldots$$

and the divergence theorem over the following differential volume;



only the surface at $\theta_2 = 1$ survives; hence

$$\oint_{C_1} d\ell \cdot \nabla^{(2)}\nu = \frac{d}{dV}\int_{S_2} dS \cdot \nabla V \times \nabla^{(2)}\nu \ .$$

The same steps may be followed to show that

$$\oint_{C_2} d\ell \cdot \nabla^{(2)}\nu = -\frac{d}{dV}\int_{S_1} dS \cdot \nabla V \times \nabla^{(2)}\nu \ .$$

These equations imply Eq. (9).

II Application of Hodges' Theorem

Given

$$B = \nabla V \times \nabla\, [\theta_1\dot{\psi}_2 - \theta_2\dot{\psi}_1 + \lambda]$$

$$\nabla V \cdot \nabla \times B = 0$$

$$\lambda \text{ single valued on } V = V(\theta_1, \theta_2, r)$$

Show that

$$\int d^3x \quad B^2 = \sum_{i=1}^{2} \int_0^C dV \; I_i(V) \; \dot{\psi}_i(V) \; .$$

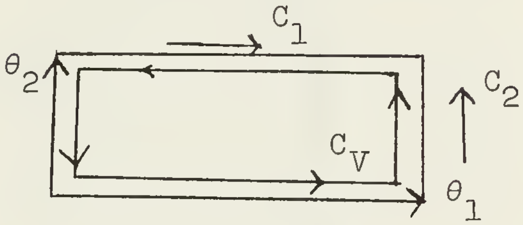
Starting with

$$\int d^3x \quad B^2 = \int d^3x \quad \underset{\sim}{B}\cdot \nabla V \times \nabla[\theta_1\dot{\psi}_2 - \theta_2\dot{\psi}_1 + \lambda]$$

we use vector identities, Eq. (14) and Stokes' theorem to show

$$\int d^3x \quad B^2 = \int_0^C dV \int ds \; \frac{\nabla V}{|\nabla V|} \cdot \nabla \times ([\theta_1\dot{\psi}_2 - \theta_2\dot{\psi}_1 + \lambda]\underset{\sim}{B})$$

$$= \int dV \oint_{C_V} d\ell \cdot B[\theta_1\dot{\psi}_2 - \theta_2\dot{\psi}_1 + \lambda]$$



With the above illustrated contours, we show

$$\int d^3 \quad B^2 = \int dV[\dot{\psi}_2 \oint_{C_2} d\ell \cdot \underset{\sim}{B} + \dot{\psi}_1 \oint_{C_1} d\ell \cdot B]$$

and hence the result.

Figure Captions

Fig. 1 The orientation of the contours C_1 and C_2 and cut surfaces S_1 and S_2 are illustrated for a given toroidal flux surface $V = V(\theta_1, \theta_2, r)$.

Fig. 2 The relation between the ordinary variation of flux as a function of volume $\delta\psi$ and the convective variation of volume as a function of flux ΔV is illustrated.

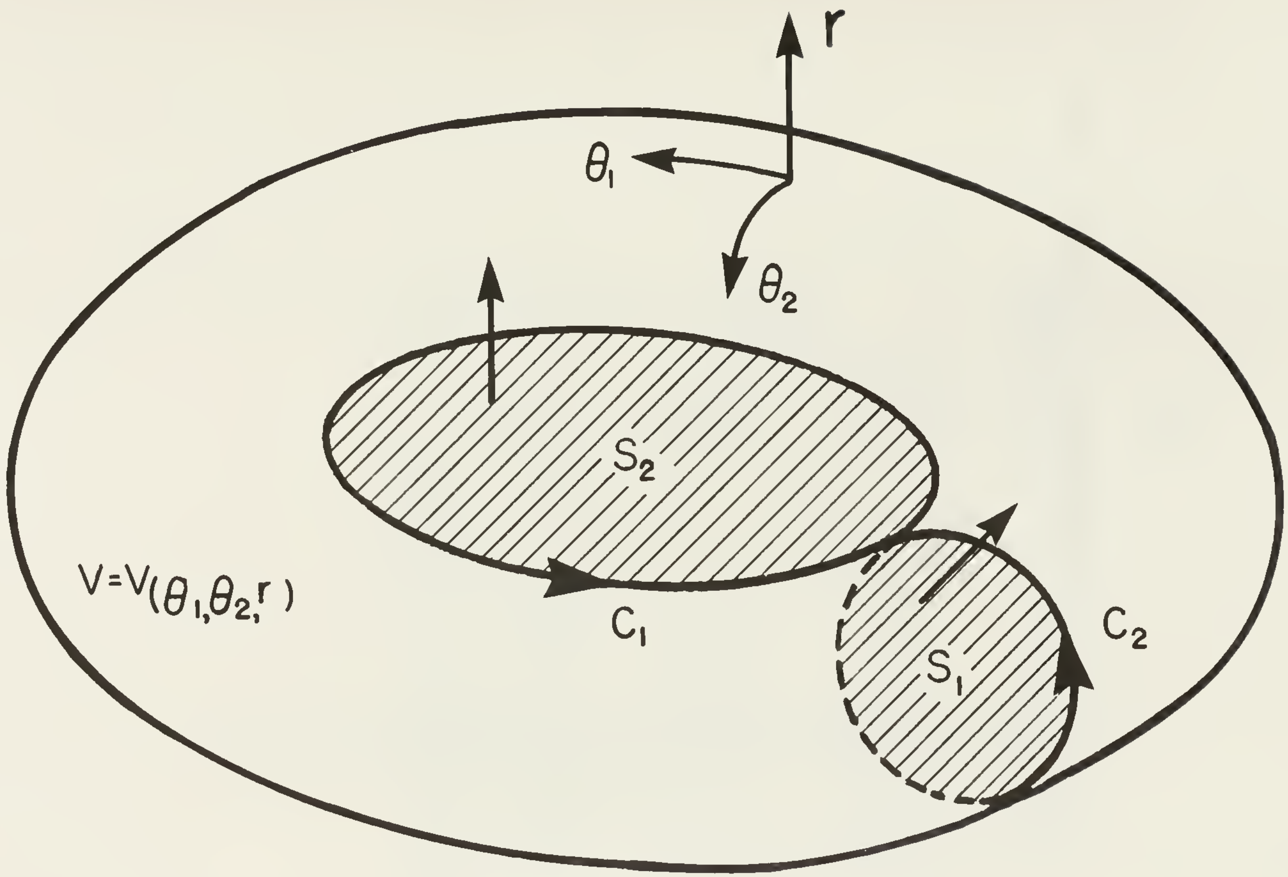


Fig. 1

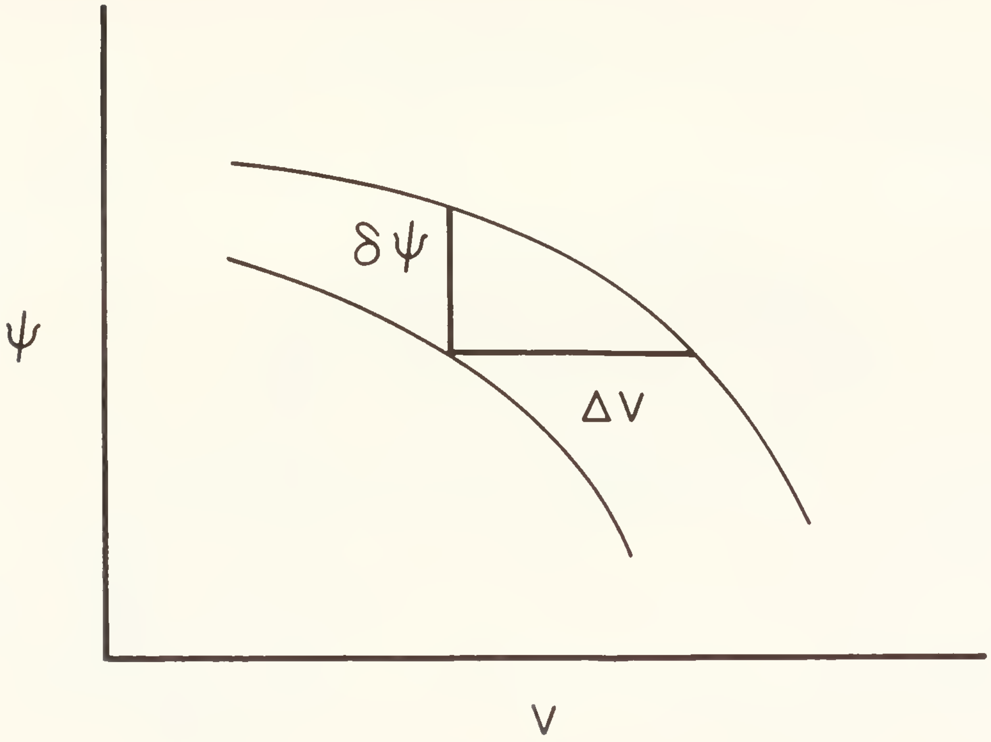


Fig. 2

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